Online Appendix: Network Methods

[**Systemic Loops and Liquidity Regulation**], by Iñaki Aldasoro and Ester Faia

This appendix presents more formal detail regarding the network measures used to characterize the simulated interbank network in the main body of the paper.[1] We consider a network with $i = 1, ..., N$ banks in which the links represent lending/borrowing relationships. As above, matrix $\mathbf{X}$ summarizes interbank connections, with element x_{ij} indicating lending from bank i to bank j. One can also define the adjacency matrix $\mathbf{A}$ with element $a_{ij} = 1$ if $x_{ij} > 0$ and $a_{ij} = 0$ otherwise. Matrix $\mathbf{A}$ therefore captures the existence of a lending/borrowing relationship without paying attention to the actual amount traded.

The *density* of a network is one of the main characteristics used to assess its topology. Density captures the proportion of actually existing connections to all possible connections that could exist in the network. Denoting by C the number of all actual connections (i.e. how many non-zero elements are present in the matrix $\mathbf{A}$) and noting that a network with N nodes can present at most $N(N-1)$ links, then network density D is given by:

$$D = \frac{C}{N^2 - N} \tag{1}$$

Using the adjacency matrix it is straightforward to define another widely used measure to characterize networks, namely the *degree* of each node. In a directed network a distinction can be made between *in-degree* (the number of incoming connections) and *out-degree* (the number of connections leaving the bank). Denoting by $\mathbf{i}$ the $N \times 1$ summation vector (i.e. a vector full of ones) and with a prime the transpose of a vector, the $N \times 1$ vector of in-degrees can be defined as $\mathbf{c}'_{in} = \mathbf{i}'\mathbf{A}$, whereas the vector of out-degrees is given by $\mathbf{c}_{out} = \mathbf{A}\mathbf{i}$. As noted in the main body of the text, the distribution of in- and out-degrees is another widely used measure to assess the topological features of a network. **??** presented the distribution of in- and out-degrees both for our simulated model and the data, showing an almost perfect match due to the method used to obtain the interbank matrix for the simulated model.

The *average degree* of a network simply computes the mean degree of either in- or out-degrees (mean out-degree and mean in-degree are obviously identical). This captures the average number of connections that can be expected for the average bank in the network and it is given by $AD = \frac{1}{N}\mathbf{i}'\mathbf{A}\mathbf{i}$.

The *shortest path length* or *geodesic* between two nodes i and j is denoted as $l(i, j)$ and as the name indicates it captures the shortest route (in number of links required) connecting i and j. There will typically be many shortest paths between a pair of nodes, but by definition they will

[1] Note that there is a small abuse of notation in this appendix. Notation used here stands alone and should not be confused with that of the main body of the paper. Unless otherwise specified, good references for the measures presented here are the comprehensive manuals by Newman (2010) and Jackson (2008).

have the same length. The *average path length* captures the ease with which one can expect to go, on average, between any pair of nodes in the network. For a directed network it is computed as

$$\bar{l} = \frac{2}{N(N-1)} \sum_{i,j \neq i} l(i,j) \tag{2}$$

Another widely used set of measures to characterize networks and rank their nodes are centrality measures. In fact, the degree measures considered above are members of this family of measures. In the main text two other measures were considered, namely *betweenness* and *eigenvector* centrality. For the two measures I report the average for all nodes in the network. Betweenness centrality is related to shortest paths: denoting by σ_{ij} the number of shortest paths between nodes i and j and by $\sigma_{i:k:j}$ the number of those paths that go through node k (with $k \neq i, j$), then the betweenness centrality of node k is given by

$$c_{BW}(k) = \sum_{i,j} \frac{\sigma_{i:k:j}}{\sigma_{ij}} \tag{3}$$

Eigenvector centrality is a global measure of centrality as it does not focus on the immediate vicinity of nodes like degree measures but considers all possible indirect connections instead. It operates under the premise that connections to nodes which are themselves well connected should be given more weight than connections to less well connected nodes. This recursive logic is aptly captured by the eigenvector method. Eigenvector centrality for all nodes in the network is then simply given by the eigenvector corresponding to the Perron eigenvalue (i.e. the largest eigenvalue) of the following eigensystem: $\mathbf{Ax} = \lambda \mathbf{x}$, where $\mathbf{x}$ denotes the eigenvector and λ the eigenvalues.

The (local) *clustering coefficient* captures the tendency of neighbors of a given node to also be connected between themselves, thereby forming a cluster of connections. Since the formal presentation of the measure would entail the introduction of further notation that is beyond the scope of this paper, the reader is referred to Barrat et al. (2004) for more formal details.

The *assortativity* coefficient in a network aims to capture the extent to which nodes with a given characteristic tend to connect to other nodes with the same characteristic. The characteristic typically used is the degree of nodes, and as seen above, in a directed network the distinction can be made between in and out degree. A positive *in-in* assortativity coefficient would for instance indicate a tendency of nodes with a high in-degree to connect to other nodes with high in-degree. Interbank networks are found to be disassortative, i.e. they present negative assortativity coefficients. Following Foster et al. (2010), one can define the set of four assortativity measures considered here using the Pearson correlation as follows:

$$s(\alpha, \beta) = \frac{C^{-1} \sum_k [(c^k_{\alpha,i} - \bar{c_\alpha})(c^k_{\beta,j} - \bar{c_\beta})]}{\sqrt{C^{-1} \sum_k (c^k_{\alpha,i} - \bar{c_\alpha})^2} \sqrt{C^{-1} (\sum_k c^k_{\beta,j} - \bar{c_\beta})^2}} \tag{4}$$

where $\alpha, \beta \in \{in, out\}$ index the degree type considered (i.e. either in- or out-degree), $c^k_{\alpha,i}$ and $c^k_{\beta,j}$ are the $\alpha-$ and $\beta-$degree of the source (i) and target (j) node for edge k respectively. Finally, $\bar{c_\alpha} = \sum_k c^k_{\alpha,i}$ (with $\bar{c_\beta}$ defined accordingly). $s(\alpha, \beta)$ will take values between -1 and 1, with positive values indicating assortativity and negative disassortativity.

The *modularity* of a network is another important structural feature that is closely related to the partition of networks into communities or *modules*. High modularity networks present many connections *within* modules but very few *between* modules. The maximization of modularity leads to an optimal partition of the networks into communities and this is the number reported in the main text. It ranges between -1/2 and 1, with positive values indicating that the number of edges within modules is higher than the number that would be expected at random. Modularity for a directed network is defined as

$$Q = \frac{1}{C} \sum_{ij} \left[a_{ij} - \frac{c_{in,i} c_{out,j}}{C} \right] \delta_{m_i, m_j} \tag{5}$$

where as before C indicates the number of connections in the network, a_{ij} stands for an element of the adjacency matrix, $c_{in,i}$ denotes the in-degree of node i, δ_{m_i,m_j} is the Kronecker delta symbol and m_i labels the module to which node i is assigned (δ_{m_i,m_j} takes the value of 1 if nodes i and j are assigned to the same module and zero otherwise). It can be shown (see Leicht and Newman (2008)) that the problem of modularity optimization in a directed weighted network reduces to finding the leading eigenvector of the matrix $\mathbf{B} + \mathbf{B}'$, where $\mathbf{B}$ is the so-called modularity matrix, with elements $b_{ij} = a_{ij} - \frac{c_{in,i} c_{out,j}}{C}$.

The last network measure considered is *reciprocity*, which captures the proportion of links going in one direction that are reciprocated by a link going in the opposite direction. Denoting by $C^{\leftrightarrow}$ the number of links which go in both directions, then reciprocity is usually defined as

$$r = \frac{C^{\leftrightarrow}}{C} \tag{6}$$

For a network in which all (no) links are reciprocated, we have that $r = 1$ ($r = 0$). This measure can be interpreted as the probability that a link is reciprocated, which both for the data and the model presented in the main text is quite high. What this measure does not allow for is the interpretation of whether the reciprocity observed is higher or lower than would be expected at random for a network of the same number of nodes and links. To this end a generalized measure proposed by Garlaschelli and Loffredo (2004) is also presented. This modified version of reciprocity is defined as follows:

$$\rho = \frac{\sum_{i \neq j} (a_{ij} - \bar{a})(a_{ji} - \bar{a})}{\sum_{i \neq j} (a_{ij} - \bar{a})^2} = \frac{r - \bar{a}}{1 - \bar{a}} \tag{7}$$

where $\bar{a} = \sum_{i \neq j} \frac{a_{ij}}{N(N-1)} = \frac{C}{N(N-1)}$ is a measure of the ratio between actual to potential links (i.e. the density). This normalized measure is akin to a correlation coefficient and allows for an interpretation similar to that of the assortativity coefficient: networks with $\rho > 0$ (< 0) are referred to as reciprocal (antireciprocal). Both networks considered in the main body of the paper are reciprocal, implying that there are more reciprocated links than could be expected at random.

References

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